The Orb Graph

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Abstract

A multi-parameter graph is an n-to-1 which has disjoint sets of edge weights which can vary independently.

The orb graph is a multi-parameter network on thirty-five vertices, and is a 6 to 1 graph.

In this paper, we give a construction and parametrization of the orb graph. We show that it has at most six solutions, and we give an example of response matrix entries for which the network is 6-to-1.
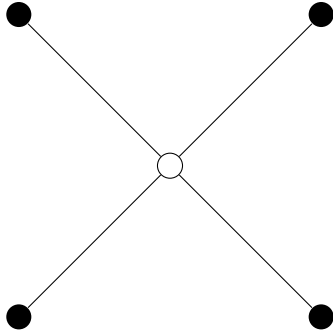
Figure 1: A 4-star

1 Basic Definitions

When we refer to a *graph*, we mean an undirected graph on n vertices. A *network* on a graph G, denoted $\Gamma(G, \gamma)$ is given by taking a two-set partition on the vertices and assigning a function γ to the edges. The vertices are split into *boundary*, denoted by ∂V, and *interior*, denoted by $int(V)$. If G has n vertices with m of them boundary, we will number first the boundary vertices 1 to m and the interior $m + 1$ to n. The function on the edges is denoted $\gamma(e)$, and assigns a positive real number to each edge.

Let $\Gamma = (G, \gamma)$ be a network on graph G with n vertices and m of them boundary. We define

$$\gamma_{i,j} = \sum \gamma(e)$$

with $\gamma_{i,j} = 0$ if there are no edges from v_i to v_j. Then the *Kirchhoff Matrix* K of Γ is given by

$$K_{i,j} = \begin{cases} \gamma_{ij} & \text{for } i \neq j \\ -\sum_{i \neq j} \gamma_{ij} & \text{for } i = j \end{cases}$$

We will generally write this matrix as a block matrix

$$K = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$$

with A an m-by-m matrix (corresponding to boundary to boundary edges), B an m-by-$(n - m)$ matrix (boundary to interior edges) and C an $(n - m)$-by-$(n - m)$ matrix (interior edges). It is fairly easy to see that by construction, this matrix is symmetric, has row sums zero and has negative real entries along the diagonal.

Now define the *response matrix* Λ as an m-by-m matrix given by

$$\Lambda = A - BC^{-}1B^T$$

This operation is also known as the *Schur complement* of the matrix K.

In simple terms, the *inverse problem* on the graph G, is as follows: Given Λ, can we find K? More exposition on the inverse problem itself can be found in [1]. It seems intuitive that there should be cases where we can find a unique solution K from the graph G and matrix and also that sometimes there are an infinite number of possible K. Ernie Esser discovered that sometimes, we can have response matrices and graphs that instead of coming from one or infinite Kirchhoff matrices, they can come from exactly n unique Kirchhoff matrices [2]. These have been dubbed n-to-1 graphs.

2 Motivation

Much previous work has been done previously concerning n-to-1 graphs. For example, see [2], [3], [4]. Generally, these graphs are constructed and parametrized with the goal of construction of a single variable polynomial of degree n whose solutions give n different possible sets of conductances that fulfill desired properties and correspond to the same response matrix. Many of these papers have described methods of constructing networks that depend on more than one independent variable, resulting in multivariate polynomials. Some very simple of a network such as this are grids that roughly correspond to the torus and give 2^n to 1 graphs. This paper seeks to consider a more interesting graph whose construction can give a 3! to 1 matrix and conjectures that using a larger central star and more arms in each claw we can construct $n!$ to 1 networks.

3 Preliminary Notions

This section gives a quick description of some concepts important to solving the inverse problem in general and which are essential to constructing and solving n to 1 graphs.

3.1 Stars and the Star-K Transformation

An *n-star* is a structure in a network constructed with a single interior vertex connected to n boundary vertices.1 I HAVE TO SAY SOMETING ABOUT COLORING OF VERTICES SOMEWHERE.

The *Star-K Transformation* takes an n-star and gives a complete graph on n vertices. This transformation is important because it lets us reconstruct original conductivities γ from new conductivities μ. For more explanation and proof on why this is true, see [3]. If we let a, b be boundary vertices connected in a star to interior vertex n, the relation between the conductivies of the start and the transformed graph is given by

$$\mu_{a,b} = \frac{\gamma_{a,n}\gamma_{b,n}}{\sigma}$$

where σ is the sum of all conductivities $\gamma_{i,n}$ in the star connected to vertex n. This was first discovered in [5].

3.2 The Quadrilateral Rule

The *Quadrilateral Rule* states that for a transformed star, the product of conductivities of two sides is equal to the product of the conductivities of the two other sides. As stated by Kempton,

$$\mu_{i,j}\mu_{k,l} = \mu_{i,k}\mu_{j,l} \text{ for all } i \neq j \neq k \neq l$$

for conductivities μ on edges between vertices i, j, k, l. Furthermore, Kempton states that a complete graph n vertices is response equivalent to an n-star if and only if it satisfies the above quadrilateral rule. For proof and further explanation, see [3], [1]. SOMEBODY ELSE?

4 Construction of the Orb Graph

To construct the orb graph, we will construct three identical "claw" pieces and connect them at three locations, which will form three triple edges between the claws, giving three equations in three parameters.

We start with a six-star, designating four vertices as "outer" and two as "inner." We add boundary edges to the outer vertices, as shown by the red dashed lines in the diagram. Then, we attach 4-stars to each pair of vertices as shown. We will call the four star attached to the inner vertices the inner arm and the other two as left arm and right arm when oriented as follows IN FIGURE. For this paper, we will number according to this diagram, being sure to number the left and right vertices as 20,21 and 22,23 respectively:
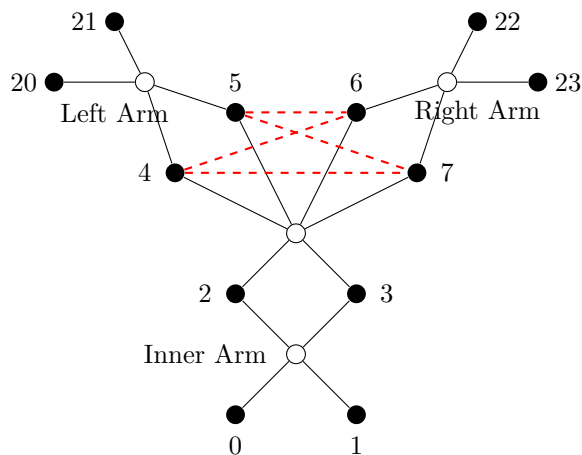

Now, copy the claw twice and label starting from 8 in the first and 13 in the second. Connect the inner arms to inner arms, left arms to left arms, and right arms to right arms, making a sort of knotted orb. When we take the star-K transformation of the graph, we will get double edges between stars, double edges at boundary to boundary edges (to assure ourselves that the graph is not immediately recoverable), and triple edges where the claws connect.
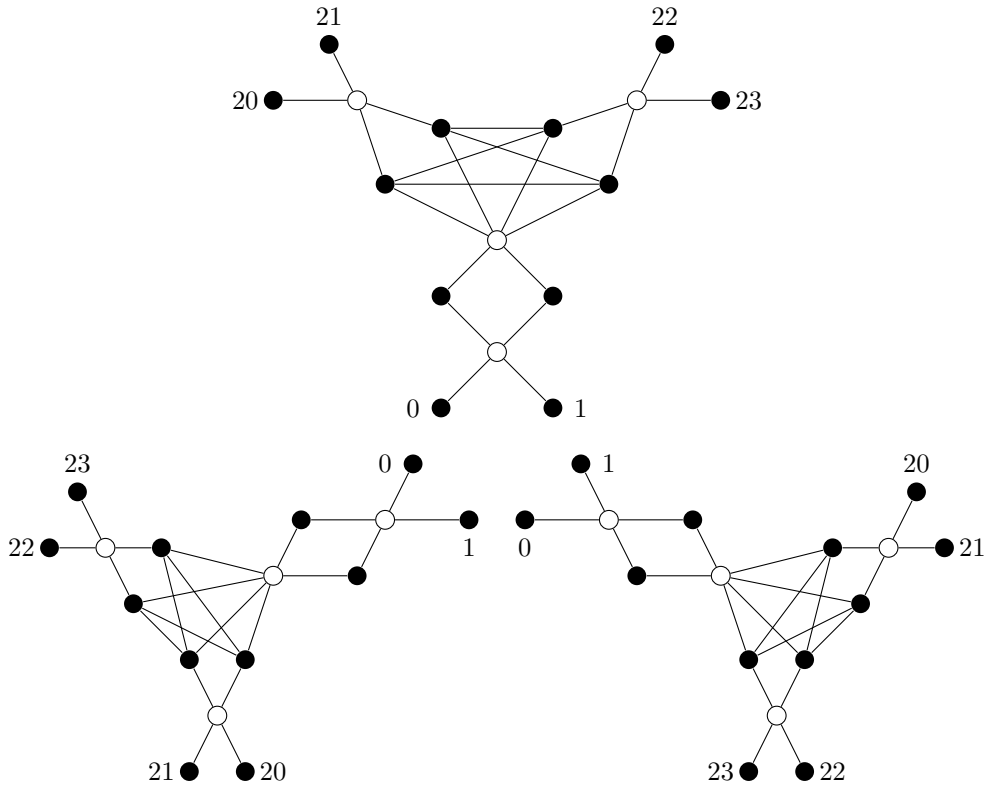
5   Parametrization of Orb Graph

First, note that the values of the transformed graph where there are only single edges between vertices will be directly found encoded in the response matrix. The double and triple edges, however, are more difficult. For example, in the $(2, 3)$ entry of the response matrix, we will find the sum of the transformed quadrilateral rule preserving conductivities of both the inner 4-star and the center 6-star. Since we only see the sum of the two conductivities, the individual conductivities are hidden and it is not clear whether we can recover the original conductivities from the sum. Since there are triple edges at $(0, 1)$, $(20, 21)$ and $(22, 23)$, this problem is compounded. These three triple edges allow the conductivities on each claw to vary independently, so we'll begin by assigning x, y and z to one of each side of the triple edge. The parameterization of the north claw will continue as follows:
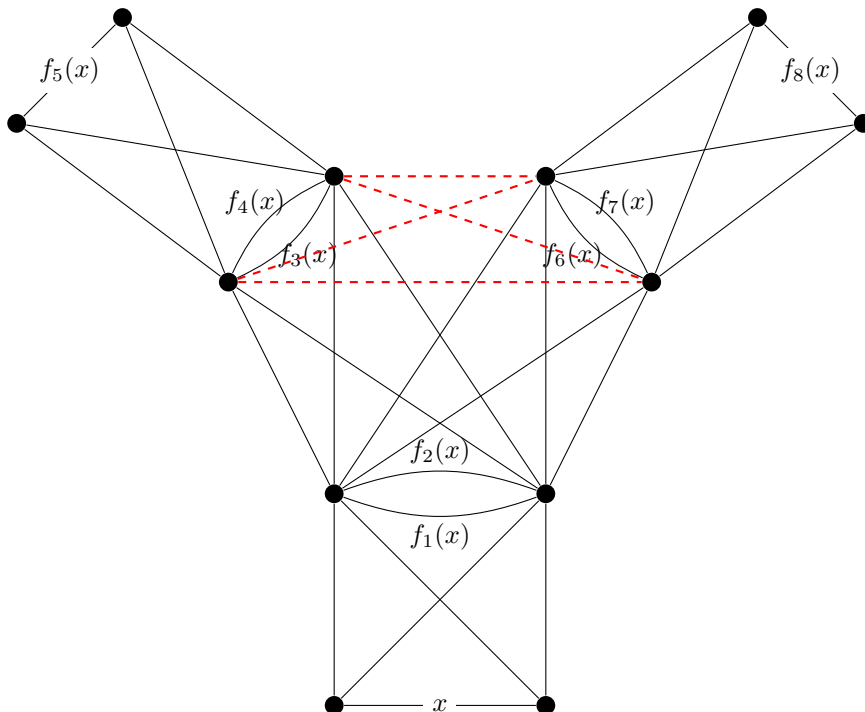
The function values are given as follows:

$$f_1(x) = \frac{\lambda_{0,2}\lambda_{1,3}}{x}$$

$$f_2(x) = \lambda_{2,3} - f_1(x)$$

$$f_3(x) = \frac{\lambda_{3,5}\lambda_{2,4}}{f_2(x)}$$

$$f_4(x) = \lambda_{4,5} - f_3(x)$$

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$$f_5(x) = \frac{\lambda_{4,20}\lambda_{5,21}}{f_4(x)}$$

$$f_6(x) = \frac{\lambda_{2,6}\lambda_{3,7}}{f_2(x)}$$

$$f_7(x) = \lambda_{6,7} - f_6(x)$$

$$f_8(x) = \frac{\lambda_{6,22}\lambda_{7,23}}{f_7(x)}$$

These parametrizations preserve values given in the response matrix as well as the quadrilateral rule. This method of parametrization was developed by [2]. For a very clear use, see [3].

Now, simply assign y to the west claw and z to the east claw, and assign each $g_i(y)$ and $h_i(z)$ symmetrically by changing the indices of the response matrix entries to correspond to the correct claw.

By writing the sums

$$\lambda_{0,1} = x + y + z$$

$$\lambda_{20,21} = f_5(x) + g_5(y) + h_5(z)$$

$$\lambda_{20,21} = f_8(x) + g_8(y) + h_8(z)$$

we have three equations in three variables and can solve for values of x, y and z.

We will add some definitions for convenience.

Definition. Let (a, b, c) be a solution set for the system of three equations given at the end of Section 5. We call (a, b, c) a *network solution on the orb graph* if $f_i(a) > 0$, $g_i(b) > 0$, and $h_i(c) > 0$ for our equations defined in Section 5.

This will be referred to later as simply a *network solution* for conciseness.

Note that the orb network is 6-to-1 if and only if we have 6 unique network solutions.

Definition. The *symmetric orb network* is the network on the orb graph where conductivites are symmetric across claws. That is, if we superimposed the three claws, aligning inner, left and right arms, edges atop each other would have the same conductivity.

6 Number of Solutions of Orb Graph

We want to find an upper bound on the number of solutions of the orb graph.

SHOULD THIS BE A THEOREM? IF SO, HOW SHOULD I STATE IT? **Theorem**

A network on the orb graph is at most 6 to 1.

Proof. Consider our system of equations given by the parametrization of the orb network:

(1)
$$\lambda_{0,1} = x + y + z$$

(2a)
$$\lambda_{20,21} = \frac{A_1 x + B_1}{C_1 x + D_1} + \frac{E_1 y + F_1}{G_1 y + H_1} + \frac{I_1 z + J_1}{K_1 z + L_1}$$

(2b)
$$\lambda_{22,23} = \frac{A_2 x + B_2}{C_2 x + D_2} + \frac{E_2 y + F_2}{G_2 y + H_2} + \frac{I_2 z + J_2}{K_2 z + L_2}$$

Multiply both sides of (2a) by $(C_1 x + D_1)(G_1 y + H_1)(K_1 z + L_1)$, then subtract terms on the left. Throughout this argument, we care more about the form of the polynomial than its coefficients, so we will be sloppy with notation of coefficients throughout.

Simplifying, we obtain a polynomial of the form

(3)
$$0 = a_1 xyz + a_2 xy + a_3 xz + a_4 yz + a_5 x + a_6 y + a_7 z + a_8$$

Now, we solve for z in (1) and substitute in (3).

$$0 = a_1 xy(\lambda_{0,1} - x - y) + a_2 xy + a_3 x(\lambda_{0,1} - x - y) + a_4 y(\lambda_{0,1} - x - y) + a_5 x + a_6 y + a_7(\lambda_{0,1} - x - y) + a_8$$

Again, simplifying coefficients for clarity's sake, we now obtain a polynomial of the form

(4a)
$$0 = c_1(x^2 y + xy^2) + c_2 x^2 + c_3 y^2 + c_4 xy + c_5 x + c_6 y + c_7$$

(4b)
$$0 = d_1(x^2 y + xy^2) + d_2 x^2 + d_3 y^2 + d_4 xy + c_5 x + d_6 y + d_7$$

Factoring (4a) and (4b), we obtain

$$0 = (c_1 x + c_3)y^2 + (c_1 x^2 + c_4 x + c_6)y + (c_2 x^2 + c_5 x + c_7)$$

(5b)
$$0 = (d_1 x + d_3)y^2 + (d_1 x^2 + d_4 x + d_6)y + (d_2 x^2 + d_5 x + d_7)$$

We now eliminate the y^2 term by multiplying (5a) by $(d_1 x + d_3)$ and (5b) by $(c_1 x + c_3)$ and subtracting. We obtain

(6)

$$0 = (d_1 x + d_3)((c_1 x^2 + c_4 x + c_6)y + (c_2 x^2 + c_5 x + c_7)) - (c_1 x + c_3)((d_1 x^2 + d_4 x + d_6)y + (d_2 x^2 + d_5 x + d_7))$$

$$= ((d_1 x + d_3)(c_1 x^2 + c_4 x + c_6) - (c_1 x + c_3)(d_1 x^2 + d_4 x + d_6))y + (d_1 x + d_3)(c_2 x^2 + c_5 x + c_7) - (c_1 x + c_3)(d_2 x^2 + d_5 x + d_7)$$

$$= (d_1 c_2 - c_1 d_2)x^3 + (d_1 c_5 + d_3 c_2 - c_1 d_5 - c_3 d_2)x^2 + (d_1 c_7 + d_3 c_5 - c_1 d_7 - c_3 d_5)x + (d_3 c_7 - c_3 d_7) - ((c_1 d_4 + c_3 d_1 - d_3 c_1 - d_1 c_4)x^2 + (c_1 d_6 + \cdots$$

Solving for y,

(7)

$$y = \frac{(d_1 c_2 - c_1 d_2)x^3 + (d_1 c_5 + d_3 c_2 - c_1 d_5 - c_3 d_2)x^2 + (d_1 c_7 + d_3 c_5 - c_1 d_7 - c_3 d_5)x + (d_3 c_7 - c_3 d_7)}{(c_1 d_4 + c_3 d_1 - d_3 c_1 - d_1 c_4)x^2 + (c_1 d_6 + c_3 d_4 - d_1 c_6 - d_3 c_4)x + (c_3 d_6 - d_3 c_6)} = \frac{p(x)}{q(x)}$$

So for most coefficients, $p(x)$ is a cubic in x and $q(x)$ is a quadratic in x.
Now, we eliminate the xy^2 and $x^2 y$ (4a) and (4b) by taking the subtraction

$$0 = d_1(c_1(x^2 y + xy^2) + c_2 x^2 + c_3 y^2 + c_4 xy + c_5 x + c_6 y + c_7) - c_1(d_1(x^2 y + xy^2) + d_2 x^2 + d_3 y^2 + d_4 xy + c_5 x + d_6 y + d_7)$$

(8)

$$= (d_1 c_2 - c_1 d_2)x^2 + (d_1 c_3 - c_1 d_3)y^2 + (d_1 c_4 - c_1 d_4)xy + (d_1 c_5 - c_1 d_5)x + (d_1 c_6 - c_1 d_6)y + (d_1 c_7 - c_1 d_7)$$

We substitute (7) for y in (8) and multiply by $q(x)^2$ to obtain

$$(d_1 c_2 - c_1 d_2)x^2 + (d_1 c_3 - c_1 d_3)\frac{p(x)^2}{q(x)^2} + (d_1 c_4 - c_1 d_4)x\frac{p(x)}{q(x)} + (d_1 c_5 - c_1 d_5)x + (d_1 c_6 - c_1 d_6)\frac{p(x)}{q(x)} + (d_1 c_7 - c_1 d_7)$$

$$0 = (d_1 c_2 - c_1 d_2)x^2 q(x)^2 + (d_1 c_3 - c_1 d_3)p(x)^2 + (d_1 c_4 - c_1 d_4)xp(x)q(x) + (d_1 c_5 - c_1 d_5)xq(x)^2 + (d_1 c_6 - c_1 d_6)p(x)q(x) + (d_1 c_7 - c_1 d_7)q(x)$$

which gives a sixth degree polynomial for most coefficients.

7 Examples

In past analysis of n to 1 graphs, the constraint equations have been simple enough to form arguments proving location and existence of network solutions that give requirements for the sets of response conductivities. Sometimes this is possible through use of derivatives and plotting functions [3] and sometimes this has been possible through the very method of construction [4]. It would be interesting to find similar conditions for the orb graph. However, since we are dealing with a polynomial of three independent variables and up to six different singularities, any discoveries in this vein will require methods more sophisticated than we now have.

Because we haven't been able to find conditions for network solutions on the orb network, instead, using Mathematica, we have been able to find examples of sets of response conductivities that have six network solutions on the orb network. This was done for symmetric cases fairly easily.

For the asymmetric case, this was much more difficult, but we were able to find solutions when left arms were symmetric and there were small variations on right arms. It turned out that in this case two of the right arms would be identical anyway. All examples found had shared properties with the symmetric case. We had three different network solution elements for z and two choices of permutable network solution elements for x and y for each z-solution. We were not able to find a solution set where there was no permutation or symmetry happening. Note that we will not give the full response matrix. We'll only give the "important" values, then the six solution sets.

North		West		East	
$\lambda_{0,2}$	1	$\lambda_{0,8}$	1	$\lambda_{0,14}$	1
$\lambda_{1,3}$	1	$\lambda_{1,9}$	1	$\lambda_{1,15}$	1
$\lambda_{2,3}$	2	$\lambda_{8,9}$	2	$\lambda_{14,15}$	2
$\lambda_{2,4}$	1	$\lambda_{8,10}$	1	$\lambda_{14,16}$	1
$\lambda_{3,5}$	1	$\lambda_{9,11}$	1	$\lambda_{15,17}$	1
$\lambda_{2,6}$	1	$\lambda_{8,12}$	1	$\lambda_{14,18}$	1
$\lambda_{3,7}$	1	$\lambda_{9,13}$	1	$\lambda_{15,19}$	1
$\lambda_{4,5}$	2	$\lambda_{10,11}$	2	$\lambda_{16,17}$	2
$\lambda_{6,7}$	3	$\lambda_{12,13}$	3	$\lambda_{18,19}$	7
$\lambda_{4,20}$	1	$\lambda_{10,20}$	1	$\lambda_{16,20}$	1
$\lambda_{5,21}$	1	$\lambda_{11,21}$	1	$\lambda_{17,21}$	1
$\lambda_{6,22}$	1	$\lambda_{12,22}$	1	$\lambda_{18,22}$	2
$\lambda_{7,23}$	1	$\lambda_{13,23}$	1	$\lambda_{19,23}$	2

$$\lambda_{0,1} = 3, \lambda_{20,21} = 8, \lambda_{22,23} = 2$$

The omitted values are unconnected vertices, which are simply zero entries, superfluous boundary-boundary double edges, and other superfluous edges. The boundary-boundary double edges and others can be assigned in any way as long as they still satisfy the quadrilateral rule.

These values give the equations

$$3 = x + y + z$$

$$8 = \frac{2x - 1}{3x - 2} + \frac{2y - 1}{3y - 2} + \frac{2z - 1}{3z - 2}$$

$$2 = \frac{2x - 1}{5x - 3} + \frac{2y - 1}{5y - 3} + \frac{8z - 4}{13z - 7}$$

and solution sets with approximate decimal values (1.03578, .686806, 1.27741), (.686806, 1.03578, 1.27741), (1.50789, .688644, .803464), (.688644, 1.50789, .803464), (.697443, 1.58692, .715636), (1.58692, .697443, .715636), which give us six possible Kirchhoff matrices.

8 Further Work

Some ideas for further work include:

- Finding conditions that give six to one graphs that satisfy the positivity conditions we need. Currently, we are working to use Sturm's Theorem on the symmetric orb graph to find some kind of conditions.

- Constructing a 4-clawed orb graph. This would be done by changing the center star to an 8-star, adding four 4-stars instead of 3, copying it three times and connecting the four claws in four different places. We conjecture that this should give a 4! to 1 graph.

- Using the 3 and 4 clawed orb graphs, perhaps classifying a family of $n!$ to 1 graphs.

Acknowledgements

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